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FLEXURAL PERTURBATIONS OF FREE JETS OF MAXWELL AND DOI–EDWARDS LIQUIDS

A. L. Yarin

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Flexural perturbations of high-velocity free jets of drop liquids moving in air are reinforced by the fact that the air pressure on the concave sections of the jet surface is greater than on the convex sections. The linear and nonlinear stages of development of flexural perturbations were studied in [1-5] for viscous Newtonian fluids. The effect of elastic stresses in the fluid on the growth of flexural perturbations of jets was first examined in [6], where it was assumed in an analysis of the growth of small disturbances that surface tension was constant along the jet, i.e., the investigators actually studied a tensed string. The studies [7, 8] examined the linear stage of growth of flexural perturbations of jets of Maxwell liquids. Our goal here is to analyze the dynamics of long-wave flexural perturbations of jets of viscoelastic fluids in both the linear and nonlinear stages of development. The rheological behavior of the fluid is described by two models – the phenomenological (Maxwell) model and the physical-molecular (Doi–Edwards) model. It is shown that the disturbances are oscillatory in character in the nonlinear stage of development. Meanwhile, the results of calculations performed with the Maxwell (M) and Doi–Edwards (DE) rheological models in the given problem agree with each other quantitatively as well as qualitatively.

1. We will examine a free jet of a drop liquid moving at the velocity U_0 in air. In the undisturbed state, the axis of the jet is straight, while its cross section is a circle of radius a_0. The densities of the liquid and air will be denoted by ρ and ρ_1, while the surface tension of the liquid will be denoted by α. The liquid is assumed to be viscoelastic. As usual, the relationship between the deviator of the stress tensor σ' and the kinematic and geometric parameters is determined by the rheological equation of state. From among rheological equations of state for concentrated systems in the literature, we will choose the two with the clearest physical meaning. The first, the Maxwell rheological equation, determines the deviator of the stress tensor in the form [9]

$$\boldsymbol{\sigma}'(t)=\frac{\mu}{\theta^2}\int\limits_{-\infty}^{t}d\tau\exp\left(-\frac{t-\tau}{\theta}\right)[\mathbf{B}_\tau(t)-\mathbf{g}]_{\mathbf{s}} \tag{1.1}$$

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where μ is the viscosity at zero shear velocity; θ is the relaxation time (μ/θ is the elastic modulus); $\mathbf{B}_\tau(t)$ is the Green tensor, accounting for deformation from the configuration at the moment τ to the configuration at the moment t; $\mathbf{g}$ is the metric tensor.

The physical meaning of Eq. (1.1) is clear: it describes the effects of the exponentially decaying memory of the liquid, lowering the elastic stresses in the liquid. Equation (1.1) was written for the simplest case of one relaxation time.

The second rheological equation of state, describing the behavior of concentrated solutions and melts of polymers, was obtained by Doi and Edwards [10] within the framework of molecular physics. It accounts for the presence of topological restrictions — neighboring macromolecules which actually surround a given macromolecular in the form of a tube and leave it only with the possibility of diffusion. The DE equation has the form [10, 11]

$$\sigma'(t)=\sum_{j_{\mathrm{odd}}=1}^{\infty}\sigma_j'(t)=G_0\sum_{j_{\mathrm{odd}}=1}^{\infty}\frac{8}{\pi^2 j^2\theta_j}\int_{-\infty}^{t}d\tau\exp\left(-\frac{t-\tau}{\theta_j}\right)\int\frac{d^2\mathbf{u}_0}{4\pi}\left\{\frac{[\mathbf{F}_\tau(t)\cdot\mathbf{u}_0][\mathbf{F}_\tau(t)\cdot\mathbf{u}_0]}{|\mathbf{F}_\tau(t)\cdot\mathbf{u}_0|^2}-\frac{\mathbf{g}}{3}\right\},\ \mu=\frac{\pi^2}{60}G_0\theta_1. \quad (1.2)$$

Here, summation is carried out over odd j; $\mathbf{u}_0$ is a unit vector randomly oriented in space ($\int d^2\mathbf{u}_0$ denotes averaging over all possible directions of $\mathbf{u}_0$); $\mathbf{F}_\tau(t)$ is the tensor of the gradient of strain from the configuration at the moment τ to the configuration at the moment t. The elastic modulus of the liquid G_0 and the spectrum of relaxation time θ_j are calculated by means of molecular parameters [10]. All of the molecular characteristics in the DE model have now been determined experimentally for several systems, while the predictions of the model for standard rheological flows agree satisfactorily with the experimental data [11-13]. Equation (1.2) was obtained for the characteristic times $t_0 \gtrsim \theta_1$ and strain rates $\sim\theta_1^{-1}$; if $t_1 \ll \theta_1$ or if the strain rates are much greater than θ_1^{-1}, then Eq. (1.2) is invalid, and it is necessary to consider the consequences of "rapid" relaxation processes [14]. The values of characteristic time at which Eq. (1.2) is valid lie approximately within the range $t_0 \geq 0.01$-0.1 sec for solutions of polybutadiene and polystyrene with concentrations of about 10-30%.

2. As is known [15], jets of viscoelastic fluids may be characterized by longitudinal surface tension (in general, relaxing). To account for the possible existence of longitudinal surface tension in a jet undergoing bending, we will assume in the general case that in the previous history of the jet ($-\infty\leqslant\tau\leqslant t$ $(t>0)$), flexural perturbations developed in the presence of tension of the jet along its axis at $-\infty\leqslant\tau\leqslant 0$. Thus, at $t > 0$, the jet will be bent with longitudinal tension. The tensor of the strain gradient

$$\mathbf{F}_\tau(t)=\mathbf{F}_\tau^*(t)\cdot\mathbf{F}_\tau^0(t). \quad (2.1)$$

Henceforth, the asterisk denotes tensors which describe bending deformation, while the zero denotes deformation by axial tension.

At $t > 0$

$$\mathbf{F}_\tau^0(t)=\mathbf{F}_\tau^0(0),\ \tau<0;\ \mathbf{F}_\tau^0(t)=\mathbf{g},\ \tau>0. \quad (2.2)$$

The Green tensor is represented as follows with allowance for (2.1)

$$\mathbf{B}_\tau(t)=\mathbf{F}_\tau(t)\cdot\mathbf{F}_\tau^{\mathrm{T}}(t)=\mathbf{F}_\tau^*(t)\cdot\mathbf{B}_\tau^0(t)\cdot\mathbf{F}_\tau^{*\mathrm{T}}(t). \quad (2.3)$$

By using (2.2), we obtain the following from (2.3) at $t > 0$

$$\begin{aligned}\mathbf{B}_\tau(t)&=\mathbf{F}_\tau^*(t)\cdot\mathbf{B}_\tau^0(0)\cdot\mathbf{F}_\tau^{*\mathrm{T}}(t),\quad \tau<0;\\ \mathbf{B}_\tau(t)&=\mathbf{F}_\tau^*(t)\cdot\mathbf{F}_\tau^{*\mathrm{T}}(t)=\mathbf{B}_\tau^*(t),\quad \tau>0.\end{aligned} \quad (2.4)$$

Inserting (2.4) into (1.1), we have the following at $t > 0$

$$\sigma'(t)=\frac{\mu}{\theta^2}\int_{-\infty}^{0}d\tau\exp\left(-\frac{t-\tau}{\theta}\right)[\mathbf{F}_\tau^*(t)\cdot\mathbf{B}_\tau^0(0)\cdot\mathbf{F}_\tau^{*\mathrm{T}}(t)-\mathbf{g}]+\frac{\mu}{\theta^2}\int_0^t d\tau\exp\left(-\frac{t-\tau}{\theta}\right)[\mathbf{B}_\tau^*(t)-\mathbf{g}]. \quad (2.5)$$

Let us isolate the part $\mathbf{A}_\tau(t)$ in the bending tensor of the strain gradient at $t > 0$, $\tau < 0$. This part of the tensor is due to the increase in flexural perturbations. Accordingly, we find

$$\mathbf{F}_\tau^*(t)=\mathbf{g}+\mathbf{A}_\tau(t),\ \mathbf{B}_\tau^*(t)-\mathbf{g}=\mathbf{A}_\tau(t)+\mathbf{A}_\tau^{\mathrm{T}}(t)+\mathbf{A}_\tau(t)\cdot\mathbf{A}_\tau^{\mathrm{T}}(t). \quad (2.6)$$

Thus, by means of the first equation of (2.6) we obtain

$$\int_{-\infty}^{0} d\tau \exp\left(-\frac{t-\tau}{\theta}\right)\left[\mathbf{F}_\tau^*(0)\cdot\mathbf{B}_\tau^0(0)\cdot\mathbf{F}_\tau^{*\mathrm{T}}(t)-\mathbf{g}\right] = \int_{-\infty}^{0} d\tau \exp\left(-\frac{t-\tau}{\theta}\right)\{[\mathbf{B}_\tau^0(0)-\mathbf{g}] + \mathbf{A}_\tau(t)\cdot[\mathbf{B}_\tau^0(0)-\mathbf{g}] +$$

$$+ [\mathbf{B}_\tau^0(0)-\mathbf{g}]\cdot\mathbf{A}_\tau^{\mathrm{T}}(t) + \mathbf{A}_\tau(t)\cdot[\mathbf{B}_\tau^0(0)-\mathbf{g}]\cdot\mathbf{A}_\tau^{\mathrm{T}}(t) + \mathbf{A}_\tau(t) + \mathbf{A}_\tau^{\mathrm{T}}(t) + \mathbf{A}_\tau(t)\cdot\mathbf{A}_\tau^{\mathrm{T}}(t)\}. \qquad (2.7)$$

As a result, with allowance for (2.7) and the second equation of (2.6), we change the expression for the stresses (2.5) to the form

$$\boldsymbol{\sigma}'(t) = \frac{\mu}{\theta^2}\int_{-\infty}^{0} d\tau \exp\left(-\frac{t-\tau}{\theta}\right)\{[\mathbf{B}_\tau^0(0)-\mathbf{g}] + \mathbf{A}_\tau(t)\cdot[\mathbf{B}_\tau^0(0)-\mathbf{g}] +$$

$$+ [\mathbf{B}_\tau^0(0)-\mathbf{g}]\cdot\mathbf{A}_\tau^{\mathrm{T}}(t) + \mathbf{A}_\tau(t)\cdot[\mathbf{B}_\tau^0(0)-\mathbf{g}]\cdot\mathbf{A}_\tau^{\mathrm{T}}(t)\} + \frac{\mu}{\theta^2}\int_{-\infty}^{t} d\tau \exp\left(-\frac{t-\tau}{\theta}\right)[\mathbf{B}_\tau^*(t)-\mathbf{g}], \qquad (2.8)$$

where the first integral describes the contribution of axial tension of the jet to the deviator stresses in the jet. Since the bending strains at the long-wave limit are sufficiently small to permit us to ignore the tensor components $\mathbf{A}_\tau(t)$ compared to unity, we obtain from (2.8)

$$\boldsymbol{\sigma}'(t) = \boldsymbol{\sigma}'^0(t) + \boldsymbol{\sigma}'^*(t) = \frac{\mu}{\theta^2}\int_{-\infty}^{0} d\tau \exp\left(-\frac{t-\tau}{\theta}\right)[\mathbf{B}_\tau^0(0)-\mathbf{g}] + \frac{\mu}{\theta^2}\int_{-\infty}^{t} d\tau \exp\left(-\frac{t-\tau}{\theta}\right)[\mathbf{B}_\tau^*(t)-\mathbf{g}]. \qquad (2.9)$$

Thus, in accordance with (2.9), the deviator stresses at $t > 0$ can be represented as the sum of the preliminary relaxing stress $\sigma'^0(t)$ and the purely bending component described by the integral term:

$$\boldsymbol{\sigma}'(t) = \boldsymbol{\sigma}'^0(t) + \frac{\mu}{\theta^2}\int_{-\infty}^{t} d\tau \exp\left(-\frac{t-\tau}{\theta}\right)[\mathbf{B}_\tau^*(t)-\mathbf{g}],$$

$$\boldsymbol{\sigma}'^0(t) = \exp(-t/\theta)\boldsymbol{\sigma}'^0(0). \qquad (2.10)$$

A similar examination for the DE liquid (1.2) leads to the expression

$$\boldsymbol{\sigma}'(t) = \boldsymbol{\sigma}'^0(t) + G_0 \sum_{j_{\text{odd}}=1}^{\infty} \frac{8}{\pi^2 j^2 \theta_j} \int_{-\infty}^{t} d\tau \exp\left(-\frac{t-\tau}{\theta_j}\right) \int \frac{d^2\mathbf{u}_0}{4\pi}\left\{\frac{[\mathbf{F}_\tau^*(t)\cdot\mathbf{u}_0][\mathbf{F}_\tau^*(t)\cdot\mathbf{u}_0]}{|\mathbf{F}_\tau^*(t)\cdot\mathbf{u}_0|^2} - \frac{\mathbf{g}}{3}\right\},$$

$$\boldsymbol{\sigma}'^0(t) = G_0 \sum_{j_{\text{odd}}=1}^{\infty} \frac{8}{\pi^2 j^2 \theta_j} \int_{-\infty}^{0} d\tau \exp\left(-\frac{t-\tau}{\theta_j}\right) \int \frac{d^2\mathbf{u}_0}{4\pi}\left\{\frac{[\mathbf{F}_\tau^0(0)\cdot\mathbf{u}_0][\mathbf{F}_\tau^0(0)\cdot\mathbf{u}_0]}{|\mathbf{F}_\tau^0(0)\cdot\mathbf{u}_0|^2} - \frac{\mathbf{g}}{3}\right\} = \sum_{j_{\text{odd}}=1}^{\infty} \boldsymbol{\sigma}_j'^0(0)\exp(-t/\theta_j). \qquad (2.11)$$

As usual, the stress tensor in the viscoelastic fluid is represented in the form $\boldsymbol{\sigma} = -p\mathbf{g} + \boldsymbol{\sigma}'$, where p is pressure.

3. As in [1, 3], we will study the dynamics of the growth of long-wave disturbances on the basis of the energy balance. If we ignore friction against the air and drag, we can approximately represent a planar flexural perturbation of the jet axis in the form

$$H = A(t)\sin(\chi s/a_0), \qquad (3.1)$$

as long as the amplitude A(t) is not too large. Here, we adopted the notation: χ is the dimensionless wave number ($\chi = 2\pi a_0/l_1$, l_1 is the wavelength of the perturbation); s is a coordinate reckoned along the axis of the undisturbed jet. The s axis is "frozen" in the moving undisturbed jet.

The work of buoyancy [1, 3, 5], leading to an increase in the bending perturbations, is expended on the increment in kinetic and surface energies and the work of internal forces. The energy balance is made up for the segment of the jet $0 \leqslant s \leqslant \pi a_0/\chi$. The following are the expressions for the work L of buoyancy during the time dt and the increments in kinetic ΔE and surface ΔE_1 energy, which obviously are independent of the rheological behavior of the fluid [1, 3]

$$L = \rho_1 U_0^2 f_0 A' A \frac{\pi\chi}{2a_0} dt, \quad \Delta E = \rho f_0 A'' A' \frac{\pi a_0}{2\chi} dt, \quad \Delta E_1 = \pi^2 \alpha A A' \frac{\chi}{2} dt, \qquad (3.2)$$

where $f_0 = \pi a_0^2$ is the cross-sectional area of the undisturbed jet; the prime denotes derivatives with respect to t.

It remains for us to calculate the work of internal forces in an isolated element of the jet during the time dt [1, 3], this work being determined by the rheological behavior of the fluid:

$$L_1 = \left\{ \int\limits_0^{\pi a_0/\chi} \left[\int\limits_{D_1} (\sigma_{ll} D_{ll} + \sigma_{nn} D_{nn} + \sigma_{bb} D_{bb})(1 - ky)\, dS \right] \lambda ds \right\} dt. \quad (3.3)$$

Here, $\sigma_{\ell\ell}$, σ_{nn}, σ_{bb} are components of the stress tensor σ (the subscript ℓ correspond to the unit vector of a tangent to the axis of the bent jet **l**; n corresponds to the unit vector of a normal **n**; b corresponds to the unit vector of a binormal **b**); $D_{\ell\ell}$, D_{nn}, D_{bb} are components of the tensor of the velocities of the bending strains **D**; k is the curvature of the jet axis; $k = -H_{,ss}/(1 + H_{,s}^2)^{3/2}$; the coordinate y is reckoned from the center of the jet cross-section D_1 along the normal; $\lambda = (1 + H_{,s}^2)^{1/2}$.

At the long-wave limit (χ < 1), according to [1, 3]

$$D_{ll} = \lambda^{-1}\lambda_{,t} - ky\omega_{,t}/\omega, \quad D_{nn} = D_{bb} = -D_{ll}/2, \quad \omega = -H_{,ss} ds/(1 + H_{,s}^2). \quad (3.4)$$

It is easy to see that in the jet

$$\sigma_{ll} D_{ll} + \sigma_{nn} D_{nn} + \sigma_{bb} D_{bb} = D_{ll}(\sigma'_{ll} - \sigma'_{nn}). \quad (3.5)$$

To calculate the integral in (3.3) with allowance for the rheology of the viscoelastic fluid (2.10) or (2.11), it is necessary to determine the tensor $\mathbf{F}_\tau^*(t)$ satisfying the differential equation

$$\frac{d\mathbf{F}_\tau^*(t)}{dt} = \nabla \mathbf{v}(t) \cdot \mathbf{F}_\tau^*(t) \quad (3.6)$$

with the initial condition $\mathbf{F}_\tau^*(\tau) = \mathbf{g}$.

As was shown in [1, 5], if we ignore the friction of air against the surface of a free jet of a viscoelastic drop liquid in the case of long-wave perturbations, the stresses $\sigma_{n\ell}$ and $\sigma_{b\ell}$ in the jet are negligible compared to the axial stress σ_{ll}: $\sigma_{nl} = O(\varepsilon\sigma_{ll})$, $\sigma_{bl} = O(\varepsilon\sigma_{ll})$, $\varepsilon \sim a_0/l_1 \ll 1$. The smallness of the stresses $\sigma_{n\ell}$ and $\sigma_{b\ell}$ indicates that the liquid section of the jet remains planar during its bending. The equality $\sigma_{nn} = \sigma_{bb}$ in the jet is a consequence of the boundary conditions for the stresses on the jet surface, this equality indicating that the deformation of the liquid section in mainly isotropic. Thus, an imaginary liquid disk cut from a jet of circular cross-section subjected to bending rotates and stretches along the axis of the jet but retains its circular cross-section. This pattern does not change with an increase in the amplitude of the perturbations as long as the motion is long-wave in character and no local sections of large curvature develop on the jet. Such sections develop only for very large perturbation amplitudes [1]. All this suggests that the increase in both infinitely small and in finite, long-wave flexural perturbations of jets of high-viscosity liquids moving in air correspond to uniaxial tension of the jets in the coordinate system connected with the jet axis. As a result, the tensors $\mathbf{F}_\tau^*(t)$, $\nabla\mathbf{v}$, and **D** are diagonal in the basis **n**, **b**, **l**. Accordingly, we can represent the tensor $\mathbf{F}_\tau^*(t)$ (as $\nabla\mathbf{v}$ and **D**) in the form $\mathbf{F}_\tau^*(t) = \mathbf{ll} f_{ll} + (\mathbf{nn} + \mathbf{bb}) f_{nn}$ and, by virtue of the diagonality, assume that $\nabla\mathbf{v} = \mathbf{D}$. Consequently, by integrating (3.6) with allowance for (3.1) and (3.4), we find the components of the tensor $\mathbf{F}_\tau^*(t)$:

$$f_{ll} = 1 + y\frac{\partial^2}{\partial s^2}[H(s,t) - H(s,\tau)] + \frac{1}{2}\left\{\left[\frac{\partial H(s,t)}{\partial s}\right]^2 - \left[\frac{\partial H(s,\tau)}{\partial s}\right]^2\right\} +$$
$$+ \frac{y^2}{2}\left[\frac{\partial^2 H(s,t)}{\partial s^2} - \frac{\partial^2 H(s,\tau)}{\partial s^2}\right]^2 + \left(\frac{\chi}{a_0}\right)^2 \sin^2\left(\frac{\chi s}{a_0}\right)\left[\frac{A^2(t) + A^2(\tau)}{2} - A(t)A(\tau)\right] + O(A^3),$$

$$f_{nn} = 1 - \frac{y}{2}\frac{\partial^2}{\partial s^2}[H(s,t) - H(s,\tau)] - \frac{1}{4}\left\{\left[\frac{\partial H(s,t)}{\partial s}\right]^2 -\right.$$
$$\left. - \left[\frac{\partial H(s,\tau)}{\partial s}\right]^2\right\} + \frac{y^2}{8}\left[\frac{\partial^2 H(s,t)}{\partial s^2} - \frac{\partial^2 H(s,\tau)}{\partial s^2}\right]^2 - \frac{1}{2}\left(\frac{\chi}{a_0}\right)^2 \sin^2\left(\frac{\chi s}{a_0}\right)\left[\frac{A^2(t) + A^2(\tau)}{2} - A(t)A(\tau)\right] + O(A^3).$$

It should be noted that in integrating (3.6), we assumed the following, with an accuracy sufficient for subsequent calculations

$$\frac{dF_\tau^*(t)}{dt} = \mathbf{ll}\frac{\partial f_{ll}}{\partial t} + (\mathbf{nn}+\mathbf{bb})\frac{\partial f_{nn}}{\partial t} + v_n\frac{\partial}{\partial y}[\mathbf{ll}f_{ll} + (\mathbf{nn}+\mathbf{bb})f_{nn}],\ v_n = \frac{\partial H}{\partial t}.$$

Using (3.7) to calculate the Green tensor for bending $\mathbf{B}_\tau^*(t) = \mathbf{F}_\tau^*(t)\cdot\mathbf{F}_\tau^{*\tau}(t)$ and inserting it into (2.10), we determine the difference of the components of the stress tensor deviator for the Maxwell liquid (this difference is needed to calculate L_1)

$$\sigma'_{ll}-\sigma'_{nn} = \sigma_0 + \frac{\mu}{\theta}\left\{-3y\left(\frac{\chi}{a_0}\right)^2\sin\left(\frac{\chi s}{a_0}\right)[A(t)-\psi_1(t)] + \frac{3}{2}\left(\frac{\chi}{a_0}\right)^2\cos^2\left(\frac{\chi s}{a_0}\right)[A^2(t)-\psi_2(t)] + \frac{3}{2}\left(\frac{\chi}{a_0}\right)^4 y^2\sin^2\left(\frac{\chi s}{a_0}\right)\times\right.$$
$$\left.\times[A^2(t)-2A(t)\psi_1(t)+\psi_2(t)] + \frac{3}{2}\left(\frac{\chi}{a_0}\right)^2\sin^2\left(\frac{\chi s}{a_0}\right)[A^2(t)-2A(t)\psi_1(t)+\psi_2(t)]\right\} + O(A^3), \qquad (3.8)$$

where

$$\psi_1(t) = \theta^{-1}\int_{-\infty}^{t} d\tau\exp\left(-\frac{t-\tau}{\theta}\right)A(\tau),\quad \psi_2(t) = \theta^{-1}\int_{-\infty}^{t} d\tau\exp\left(-\frac{t-\tau}{\theta}\right)A^2(\tau), \qquad (3.9)$$

while $\sigma_0 = \sigma'^0_{ll} - \sigma'^0_{nn}$ characterizes the initial surface tension of the jet.

Inserting (3.4), (3.5), and (3.8) into (3.3) and integrating with allowance for (3.1), we have

$$L_1 = \left\{\sigma_0\frac{\chi}{a_0}AA'\frac{\pi}{2}f_0 + \frac{3\mu}{\theta}I_0\left(\frac{\chi}{a_0}\right)^3\frac{\pi}{2}A'(A-\psi_1) + \right.$$
$$\left. + \frac{\mu}{\theta}f_0\frac{3}{2}\left(\frac{\chi}{a_0}\right)^3 AA'\pi\left[\frac{3}{8}(A^2-\psi_2) + \frac{1}{8}(A^2-2A\psi_1+\psi_2)\right]\right\}dt, \qquad (3.10)$$
$$I_0 = \pi a_0^4/4.$$

The higher-order terms with respect to perturbation amplitude and the terms through the thickness of the jet (long-wave approximation, thin jet) are discarded.

Using (3.2) and (3.10) to make up the energy balance $L = \Delta E + \Delta E_1 + L_1$, we obtain the following equation for the amplitude of the perturbation $A(t)$:

$$A'' + \frac{3}{4}\frac{\mu}{\rho a_0^2}\chi^4\frac{Y_1}{\theta} + \frac{3}{4}\frac{\mu}{\rho a_0^4}\chi^4\frac{A(AY_1+Y_2)}{\theta} + A\chi^2\left[\frac{\alpha}{\rho a_0^3} - \frac{\rho_1 U_0^2}{\rho a_0^2} + \frac{\sigma_0(t)}{\rho a_0^2}\right] = 0,$$
$$\sigma_0(t) = [\sigma'^0_{ll}(0) - \sigma'^0_{nn}(0)]\exp(-t/\theta), \qquad (3.11)$$

The functions $Y_1(t) = A - \psi_1$ and $Y_2(t) = A^2 - \psi_2$ entering into (3.11), with allowance for (3.9), are determined by integration of the following differential equations together with (3.11):

$$Y_1' = A' - Y_1/\theta,\ Y_2' = 2AA' - Y_2/\theta. \qquad (3.12)$$

It should be noted that if we do not make simplifications leading from (2.8) to (2.9) and we use (2.8) directly, then instead of the term $W_1 = A\chi^2\sigma_0(t)/(\rho a_0^2)$ in Eq. (3.11) we will have $W_2 = A\chi^2\sigma_0(t)[1+\Phi(A)]/(\rho a_0^2)$, $\Phi(A) \leqslant O(A)$. As it turns out, this change of terms may change the result of calculation of the nonlinear stage of bending, when there is a stress $\sigma_0(t) \neq 0$ caused by the initial longitudinal surface tension of the jet. However, as follows from [1, 3-5] and from the results of the calculations discussed in Part 4, nonlinear effects will be important only over a period of time exceeding the relaxation time for the stress $\sigma_0(t)$, when W_1 and W_2 are nearly equal to zero. This justifies the transition from (2.8) to (2.9) in the derivation of Eq. (3.11), which is nonlinear with respect to the amplitude A.

Performing similar calculations for the DE liquid, we obtain the following by means of Eqs. (2.11), (3.1), (3.4), (3.5), and (3.7)

$$A'' + \frac{6}{5\pi^2}\frac{G_0\chi^4}{\rho a_0^2}\sum_{j_{\text{odd}}=1}^{\infty}\frac{Y_{1j}}{j^2} + \frac{6}{5\pi^2}\frac{G_0\chi^4}{\rho a_0^4}A\sum_{j_{\text{odd}}=1}^{\infty}\frac{AY_{1j}+Y_{2j}}{j^2} + A\chi^2\left[\frac{\alpha}{\rho a_0^3} - \frac{\rho_1 U_0^2}{\rho a_0^2} + \frac{\sigma_0(t)}{\rho a_0^2}\right] = 0,$$
$$Y_{1j}' = A' - Y_{1j}/\theta_j,\ Y_{2j}' = 2AA' - Y_{2j}/\theta_j, \qquad (3.13)$$

where

$$Y_{1j} = A - \theta_j^{-1}\int_{-\infty}^{t} d\tau\exp\left(-\frac{t-\tau}{\theta_j}\right)A(\tau),$$

$$Y_{2j} = A^2 - \theta_j^{-1} \int_{-\infty}^{t} d\tau \exp\left(-\frac{t-\tau}{\theta_j}\right) A^2(\tau),$$
$$\sigma_0(t) = \sum_{j_{odd}=1}^{\infty} \left[\sigma_{jll}'^{0}(0) - \sigma_{jnn}'^{0}(0)\right] \exp(-t/\theta_j). \quad (3.14)$$

4. First we will examine small perturbations, when the terms in (3.11) and (3.13) which are nonlinear with respect to A [the third terms $O(A^3)$] can be ignored. In this case, $A = A_0 \exp(\gamma t)$ (γ is the increment and A_0 is the amplitude of the bending perturbation at $t = 0$). Also,

$$Y_1 = \gamma\theta A/(1 + \gamma\theta),\ Y_2 = 2\gamma\theta A^2/(1 + 2\gamma\theta), \quad (4.1)$$

and we obtain the following characteristic equation from (3.11) and (3.12) for a Maxwell liquid

$$\gamma^2 + \frac{3}{4}\frac{\mu\chi^4}{\rho a_0^2(1+\gamma\theta)}\gamma + \chi^2\left(\frac{\alpha}{\rho a_0^3} - \frac{\rho_1 U_0^2}{\rho a_0^2} + \frac{\sigma_0}{\rho a_0^2}\right) = 0. \quad (4.2)$$

Here, it is assumed that initial surface tension is either absent from the jet ($\sigma_0 = 0$) or is "frozen" and that $\sigma_0 = \text{const} \neq 0$.

Equation (4.2) generalizes the characteristic equation for a viscous Newtonian fluid [1, 3] with $\theta = 0$ to the case of a viscoelastic Maxwell liquid. It predicts (at $\sigma_0 = 0$) acceleration of the growth of small bending perturbations of a jet of a Maxwell liquid compared to a comparable jet of a Newtonian fluid (ρ, μ, a_0, α, U_0 = idem) due to a decrease in effective viscosity $\mu_e = \mu/(1 + \gamma\theta)$. Initial surface tension $\sigma_0 > 0$ is a stabilizing factor which retards the growth of the perturbations (the value $\gamma > 0$ decreases) or in general prevents the development of flexural perturbations at $(\sigma_0 + \alpha/a_0) > \rho_1 U_0^2$, when $\text{Re}\{\gamma\} < 0$. The result obtained in [6], in which a jet was represented as a tensed string, is transformed at the long-wave limit to Eq. (4.2), with $\mu = \alpha = 0$.

The solution of the problem of the bending of a jet of a Maxwell liquid in a flow of air depends on eight parameters having three independent dimensions: μ, ρ, a_0, α, ρ_1, U_0, θ, σ_0. Thus, in accordance with the π-theorem of dimensional theory, the solution is determined by five similarity criteria:

$$\Pi_1 = \frac{\rho_1}{\rho},\ \Pi_2 = \frac{\mu^2}{\rho a_0^2 \rho_1 U_0^2},\ \Pi_3 = \frac{\rho_1 U_0^2}{\mu/\theta},$$
$$\Pi_4 = \frac{\sigma_0}{\rho_1 U_0^2},\ \Pi_5 = \frac{\alpha}{\rho_1 U_0^2 a_0}. \quad (4.3)$$

Figure 1 shows the functions $\gamma(\chi)$ calculated by means of Eq. (4.2) with the following values of the parameters: for all curves $\Pi_1 = 10^{-3}$, $\Pi_4 = 0$; for 1 and 2 $\Pi_2 = 0.156 \cdot 10^4$, $\Pi_5 = 0.94 \cdot 10^{-3}$; 3 and 4 $\Pi_2 = 0.4 \cdot 10^4$, $\Pi_5 = 2.4 \cdot 10^{-3}$; 2 and 4 $\Pi_3 = 0$; 1 and 3 $\Pi_3 = 0.64$, $\Pi_3 = 0.25$. Lines 1 and 3 correspond to two jets of a Maxwell liquid differing in velocity, the velocity being greater in the first case. Lines 2 and 4 correspond to the results of 1 and 3 for a Newtonian fluid. There is a marked difference between the results in our Fig. 1 and the results in Fig. 4 of [8] (which correspond to the same parameter values). For example, the increment of γ_{max} for curve 1 is roughly 30% greater in our calculation than in [8]. This evidently makes it possible to evaluate the error introduced by the simplifications of the characteristic equation in [8]. As a result of these simplifications, the value of γ was reduced to the second order of magnitude. Line 2 in Fig. 2 shows the dimensionless function $\gamma(\chi)$ determined by characteristic equation (4.2) for a Maxwell liquid with $\Pi_1 = 10^{-3}$, $\Pi_2 = 0.4 \cdot 10^4$, $\Pi_3 = 0.25$, $\Pi_4 = \Pi_5 = 0$, while line 1 shows the result for a corresponding jet of Newtonian fluid (Π_1, Π_2, Π_4, Π_5 = idem, $\Pi_3 = 0$). The increment is referred to the below quantity T^{-1}.

The characteristic equation for small flexural perturbations of a jet of a DE liquid, when $A = A_0 \exp(\gamma t)$,

$$Y_{1j} = \gamma\theta_j A/(1 + \gamma\theta_j),\ Y_{2j} = 2\gamma\theta_j A^2/(1 + 2\gamma\theta_j), \quad (4.4)$$

is obtained from (3.13), with σ_0 = const, in the form

$$\gamma^2 + \frac{6}{5\pi^2}\frac{G_0\chi^4}{\rho a_0^2}\gamma \sum_{j_{odd}=1}^{\infty} \frac{\theta_j}{(1+\gamma\theta_j)\, j^2} + \chi^2\left(\frac{\alpha}{\rho a_0^3} - \frac{\rho_1 U_0^2}{\rho a_0^2} + \frac{\sigma_0}{\rho a_0^2}\right) = 0. \quad (4.5)$$

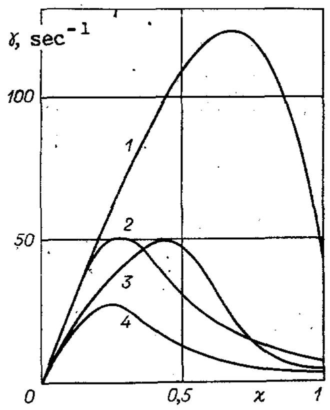

Fig. 1

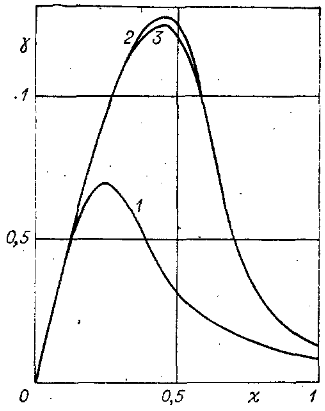

Fig. 2

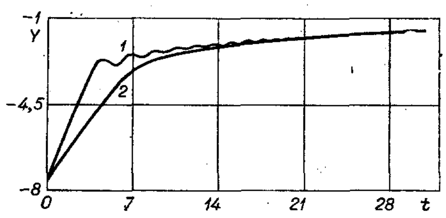

Fig. 3

With allowance for the second equation of (1.2), as before the similarity criteria will be Π_1-Π_5 from (4.3). To these criteria we add a criterion which determines the ratio of the spectral relaxation times $\Pi_m = \theta_k/\theta_p$. Curve 3 in Fig. 2 shows the relation $\gamma = \gamma(\chi)$ calculated as the solution of characteristic equation (4.5) for the same values of the criteria Π_1-Π_5 as for curve 2 and five relaxation modes (j = 1, 3, 5, 7, 9). A change in the number of modes in the calculation has a very slight effect on the result. In fact, the relation $\gamma = \gamma(\chi)$ predicted by characteristic equation (4.5) for a jet of DE liquid coincides with the result obtained in the solution of Eq. (4.2) for a jet of Maxwell liquid with the same values of Π_1-Π_5.

It should be noted that by virtue of (4.5), the initial surface tension $\sigma_0 > 0$, as for the jet of Maxwell liquid, slows the development of small flexural perturbations. At $(\sigma_0 + \alpha/a_0) > \rho_1 U_0^2$, the jet is completely stabilized.

Let us proceed to analysis of the growth of fairly large flexural perturbations, when the terms in Eqs. (3.11) and (3.13) (third terms) which are nonlinear with respect to the amplitude A are important. In the case of a Maxwell liquid, it is necessary to numerically integrate system (3.11) and (3.12) with the initial conditions

$$t = 0,\ A = A_0,\ A' = A_0\gamma_*,\ Y_1 = \gamma_*\theta A_0/(1 + \gamma_*\theta),\quad Y_2 = 2\gamma_*\theta A_0^2/(1 + 2\gamma_*\theta). \tag{4.6}$$

In (4.6), the value of the increment γ_* corresponds to the maximum γ_{max} determined by characteristic equation (4.2). Thus, in calculating the nonlinear evolution of flexural perturbations, we assume $\chi = \chi_*$. Here, the value of the dimensionless wave number χ_* corresponds to γ_{max}: $\gamma_* = \gamma_{max} = \gamma(\chi_*)$. Conditions (4.6) correspond to initially small perturbations, so that the initial conditions for functions Y_1 and Y_2 are determined by means of (4.1).

Curve 1 in Fig. 3 shows the results of calculations for a jet of a Maxwell liquid with $\Pi_1 = 10^{-3}$, $\Pi_2 = 0.4\cdot 10^4$, $\Pi_3 = 0.25$, $\Pi_4 = \Pi_5 = 0$, while curve 2 shows the result for a corresponding jet of Newtonian fluid ($\Pi_3 = 0$). In this and subsequent figures, the amplitude of the flexural perturbation of the jet of viscoelastic fluid $H_{max} = A$ is referred to the wavelength of the perturbation $l_{1*} = 2\pi a_0/\chi_*$, while the time is referred to $T = (\mu\rho a_0^2)^{1/3}/(\rho_1 U_0^2 - \alpha/a_0)^{2/3}$; $Y = \ln(H_{max}/l_{1*})$. For the Newtonian fluid, the amplitude of the perturbation H_{max} is referred to the wavelength of the perturbation $l_{2*} = 2\pi a_0/\chi_* = 2\pi a_0 \Big/ \left[\frac{8}{9}\frac{\rho a_0^2}{\mu^2}\times(\rho_1 U_0^2 - \alpha/a_0)\right]^{1/6}$ [1, 4, 5]; $Y = \ln(H_{max}/l_{2*})$. In accordance with the data in Figs. 1 and 2, $l_{2*} > l_{1*}$. The difference in Y in Fig. 3 for the viscoelastic and Newtonian fluids with a fixed value of t is evidently due not to the difference in the scales of l_{2*} and l_{1*}, but to the different rates of development of small perturbations. In the calculations, we assumed that $A_0/l_{1*} = 5\cdot 10^{-4}$, $A_0/l_{2*} = 5\cdot 10^{-4}$.

It follows from Fig. 3 that in the case of the viscoelastic Maxwell fluid – in contrast of the monotonic (slowed by stresses associated with a nonlinear effect – the elongation of the jet axis) increase in flexural perturbations of the jet of Newtonian fluid – a new nonlinear effect is seen – oscillations of the amplitudes of the perturbations. These oscillations, which result in a decrease in the amplitude of the disturbances over certain time intervals, are the result of the competition of inertial and elastic forces. The jet element undergoing bending jumps past its "equilibrium" position due to inertia, and the axis

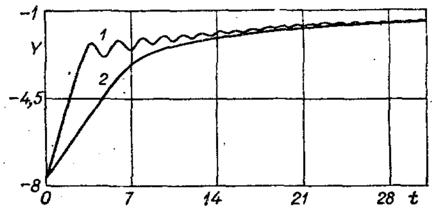

Fig. 4

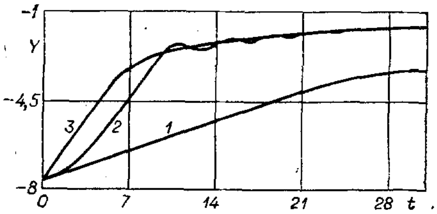

Fig. 5

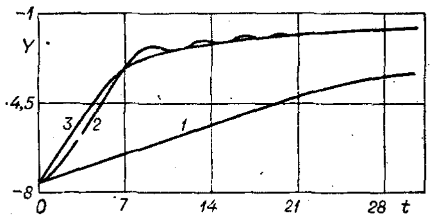

Fig. 6

of the jet is additionally elongated (due to the finiteness of the perturbations, the length of the jet axis is constant during the linear stage), which produces an additional elastic force along the element that tends to contract it. Thus, the inverse process begins: shortening of the element, a reduction in the amplitude of the flexural perturbation, a new jump past the "equilibrium" position due to inertia, etc. Viscous dissipation gradually absorbs the energy of these oscillations.

It is this very mechanism that is evidently also responsible for oscillations of the surface of a straight jet of a Maxwell liquid during capillary decay, discovered in calculations in [16], and the oscillations in the solutions of other nonsteady problems of the dynamics of viscoelastic fluids when allowance is made for their inertia.

In calculating the growth of finite flexural perturbations of a jet of DE liquid, the following initial conditions are established for system (3.13) with allowance for (4.4)

$$t = 0,\ A = A_0,\ A' = A_0\gamma_*,\ Y_{1j} = \gamma_*\theta_j A_0/(1 + \gamma_*\theta_j),$$
$$Y_{2j} = 2\gamma_*\theta_j A_0^2/(1 + 2\gamma_*\theta_j). \qquad (4.7)$$

In conditions (4.7), the increment γ_*, as the value $\chi = \chi_*$ in (3.13), is determined by the most rapidly-growing small perturbation, i.e., it corresponds to $\gamma_* = \gamma_{max}(\chi_*)$ in the solution of characteristic equation (4.5). The results of the calculation for a jet of DE liquid with $\Pi_1 = 10^{-3}$, $\Pi_2 = 0.4\cdot 10^4$, $\Pi_3 = 0.25$, $\Pi_4 = \Pi_5 = 0$ and five modes in the relaxation spectrum nearly coincides with curve 1 in Fig. 3. Line 1 in Fig. 4 shows the coincident results of calculations for jets of M and DE liquids ($j = 1,\ldots,9$) with $\Pi_1 = 10^{-3}$, $\Pi_2 = 0.156\cdot 10^4$, $\Pi_3 = 0.64$, $\Pi_4 = \Pi_5 = 0$, while line 2 shows the result for a corresponding jet of Newtonian fluid ($\Pi_3 = 0$).

The calculations performed with the M and DE rheological models show that the oscillations which accompany the nonlinear stage of growth of flexural perturbations are reinforced with an increase in the similarity criterion Π_3 while Π_i ($i \neq 3$) remain constant. Since the increase in Π_3 is equivalent to a reduction in the elastic modulus of the liquid, an increase in Π_3 is accompanied by an increase in the "compliance" of the liquid and facilitation of the inertial jump past the "equilibrium" position.

Use of the DE model, which ignores the details of "rapid" relaxation processes, is justified by the data in Figs. 3 and 4 if $t_0 \simeq 20T \gtrsim \theta_1$ and $\gamma \sim \theta_1^{-1}$. The latter condition is satisfied in the present case.

Figure 5 shows the results for an M liquid with the same values of parameters Π_1-Π_3, Π_5 as in Fig. 3 but with initial surface tension $\sigma_0(0) = \rho_1 U_0^2/1.1$. We examined two variants: with the initial stress "frozen" and remaining constant during bending of the jet (curve 1); with the initial surface tension relaxing in accordance with the second equation of (3.11) (curve 2); curve 3 shows the results for the corresponding jet of Newtonian fluid. Comparison of lines 1 and 2 with each other and with line 1 in Fig. 3 illustrates the stabilizing role of longitudinal surface tension in the process of development of perturbations. Of course, in the case of a "frozen" value of σ_0, the stabilizing effect of longitudinal surface tension during perturbation growth is considerably greater.

Figure 6 shows results corresponding to a jet of DE liquid (the values of Π_1-Π_3, Π_5 are the same as in Fig. 3, and we considered five relaxation modes) with $\sigma_0(0) = \rho_1 U_0^2/1.1$. Line 1, obtained for $\sigma_0 = \text{const}$, nearly coincides with line 1 in Fig. 5. Line 2 corresponds to the relaxation of the initial stress in accordance with the third equation of (3.14) and of course differs from line 2 in Fig. 5, since relaxation of the initial surface tension proceeds differently in jets of M liquids (single-mode model) and DE liquids (multiple-mode model); line 3 shows results for the corresponding jet of Newtonian fluid.

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PHYSICAL MODELING OF TURBULENT THERMICS

Yu. A. Gostintsev, Yu. S. Matveev, V. E. Nebogatov, and A. F. Solodovnik

UDC 536.253+532.529

In order to investigate atmospheric convection processes and the development of safe methods of exploiting and storing explodable and toxic mixtures, and a number of ecological problems, it is necessary to study nonstationary convective flows that occur when a mass of light gas rises in a field of gravitational forces (thermics). Large-scale turbulent flows are of greatest interest from the practical viewpoint since the general complexity of studying them is due to the restricted possibility of obtaining direct experimental data. In this connection, a study of the modeling laws for turbulent thermics acquires special value.

At a certain time, let there be a free volume V_0 of gas with density ρ_0 different from the density ρ_a of the environment, in open space. The convective current that occurs is due to the action of the force $F = g(\rho_a - \rho_0)V_0$, the resultant of the Archimedes and gravity forces. For currents in an unstratified medium the quantity F is conserved in time: $F = g\int_{V_\infty}(\rho_a - \rho(t))dV = g(\rho_a - \rho_0)V_0$, which is a result of the law of conservation of the excess quantity

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